

A Dynamical System Analysis of Holographic Dark Energy Models with Different IR Cutoff

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The paper deals with a dynamical system analysis of the cosmological evolution of an holographic dark energy (HDE) model interacting with dark matter (DM) which is chosen in the form of dust. The infrared cut-off of the holographic model is considered as future event horizon or Ricci length scale.

The interaction term between dark energy and dark matter is chosen of following three types

- i) proportional to the sum of the energy densities of the two dark components
- ii) proportional to the product of the matter energy densities and
- iii) proportional to dark energy density.

The dynamical equations are reduced to an autonomous system for the three cases and corresponding phase space is analyzed.

Keywords: Holographic Dark Energy, Interaction, Autonomous System, Phase plane.

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I. INTRODUCTION

In recent past, there were a series of cosmological observations particularly from Type Ia Supernovae [1-3], Cosmic Microwave Background Radiation [CMBR][4], Baryon Accoustic Oscillation [5, 6] which indicate cosmic acceleration. According to these observational evidences [7-9], at present our universe consists of approximately 25% non-baryonic gravitating matter (known as dark matter (DM)), 70% non-baryonic non gravitating unknown matter (known as dark energy (DE)) and 5% radiation and baryonic matter which is well understood by the standard models of particles. This dark matter is responsible for large scale structure formation in the universe.

Although, the nature of the dark energy is still unknown, there are some possible candidates to describe it. A natural choice for this DE is the cosmological constant which represents a vacuum energy density having constant equation of state $\omega = -1$ and this model (Λ CDM) fits well with large number of observations. But it suffers from two major drawbacks namely *fine-tuning problem* (i.e why the observed value is far below than the estimation from quantum field theory ?) and ii) *coincidence problem* (why the constant vacuum energy and matter energy densities are precisely of the same order today?) As a result of these observational [10] and theoretical [11,12] probes for the cosmological constant, dynamical DE models have been proposed in the literature. Scalar field models [12 - 15] including quintessence, K-essence, tachyon etc have attracted lots of attention. Other models of dark energy include interacting DE models, braneworld models, chaplygin gas models and many others.

Dominant contribution of the dark matter and dark energy to the matter source of the universe draws a lot of interest in studying coupling between dark matter and dark energy [16 - 32]. Further, recently it has been shown that proper choice of the non-gravitational interaction term may alleviate the coincidence problem [20 - 24].

The holographic principle asserts that the number of relevant degrees of freedom of a system dominated by gravity must vary along with the area of the surface bounding the system [25]. It is speculated in the literature that use of holographic principle [26, 27, 33] may shed some light into the unknown and mysterious nature of DE. Such a dark energy model is termed as Holographic Dark Energy (HDE) model. The HDE model is an attempt to apply holographic principle of the theory of quantum gravity to dark energy problem. Accordingly, the energy density of any given region should be bounded by that ascribed to a Schwarzschild black hole that fills the same volume [27, 33]. Mathematically, it means $\rho_d \leq M_p^2 L^{-2}$, where ρ_d is the energy density for dark energy, $M_p = (8\pi G)^{-\frac{1}{2}}$ is the reduced Planck mass and L is the size of the region (i.e IR cut off).

Usually from effective quantum field theory the DE density can be written as [26,33]

$$\rho_d = \frac{3M_p^2 c^2}{L^2} \quad (1)$$

where the dimensionless parameter 'c' is related to the uncertainties of the theory and factor 3 is for mathematical convenience. We can see that with $L = \text{constant}$, it portrays the cosmological constant model.

There are many choices for the IR cut off of which three are widely used in the literature.

i) **Hubble radius** i.e $L = H^{-1}$ but it yields a wrong equation of state for dark energy but a correct DE density which is close to the observed value [21,26].

ii) **Future event horizon** i.e $L = R_E$: M Li suggested that future event horizon should be chosen as the IR cut off to explain both the fine tuning and coincidence problem [27, 32, 34].

iii) **Ricci's scalar curvature** [28 - 31, 36] i.e $L = (\dot{H} + 2H^2)^{-\frac{1}{2}}$: the reason for choosing this length scale is that it corresponds to the size of maximal perturbation, leading to the formation of black hole. This model avoids fine tuning problem and coincidence problem. People have studied extensively both HDE and interacting HDE models at different length scale [35, 36, 37, 38, 39].

In the present work we first assume HDE model for $L = R_E$. Then we consider modified holographic dark energy model at Ricci's Scale (MHRDE)[40]. In both the cases, we have used interaction with phenomenological form. We will study the models qualitatively and will check for viable cosmological solution considering cosmological constraints and observational data. In order to study the dynamical character of the system, critical points are obtained. Any standard text will explain the method [41]. Corresponding cosmological models are analyzed. Feasible cosmological solutions should depict our present universe as global attractor i.e all the possible initial conditions lead to the observed percentages of dark energy and dark matter, once reached, they remain fixed forever [42]. For this reason we will focus on the stability of critical points i.e cosmological models which are attractors.

The paper is organized as follows: next section describes the basic equations and interaction terms of the model. Section 3 presents a phase space analysis of interacting HDE model taking future event horizon as IR cutoff for two type of interactions. Interacting MHRDE model has been considered in section 4 for phase space analysis and summary of the work has been presented in section 5.

II. BASIC EQUATIONS

The homogeneous and isotropic flat FRW spacetime is chosen as the model of the universe. It is assumed to be filled up with DM in the form of dust (having energy density ρ_m) and HDE in the form of perfect fluid with variable equation of state i.e $p_d = \omega_d \rho_d$ with ρ_d , p_d as the energy density and

thermodynamic pressure of the dark energy respectively. Dark matter and dark energy interact non gravitationally with one another.

The Einstein field equations can be written as (in the units : $8\pi G = c = 1$).

$$3H^2 = \rho_m + \rho_d \quad (2)$$

$$2\dot{H} = -\rho_m - (1 + \omega_d)\rho_d \quad (3)$$

Using field equations (2) and (3) , the acceleration of the universe is given by

$$\ddot{a} = -\frac{a}{6}[\rho_m + \rho_d(1 + 3\omega_d)] \quad (4)$$

which shows that it is required to have $\omega_d < -\frac{1}{3}$ for cosmic acceleration. Introducing the density parameters, the first Friedman equation (i.e eqn (2)) can be written as

$$\Omega_m + \Omega_d = 1 \quad (5)$$

where

$$\Omega_m = \frac{\rho_m}{3H^2} = \frac{u}{1+u} \quad \text{and} \quad \Omega_d = \frac{\rho_d}{3H^2} = \frac{1}{1+u} \quad (6)$$

with $u = \frac{\rho_m}{\rho_d}$, the ratio of energy densities. Now considering the interaction term between the two dark components the energy conservation relations take the form

$$\dot{\rho}_m + 3H\rho_m = Q \quad (7)$$

and

$$\dot{\rho}_d + 3H(1 + \omega_d)\rho_d = -Q \quad (8)$$

Here Q , the interaction term is not unique. We choose $Q > 0$ so that there is transfer of energy from DE component to DM component i.e dark energy decays into dark matter. The positivity of Q ensures the validity of the second law of thermodynamics and satisfies the Le Chatelier's principle [16]. Also $Q > 0$ is in favour of resolving the coincidence problem. It should be noted that we have not included baryonic matter in the interaction due to the constraints imposed by local gravity measurements [16, 43]. In the next sections we shall consider three different choices of interaction term separately, namely (i) $Q = 3b^2H\rho$ ($\rho = \rho_m + \rho_d$, the total energy density), ii) $Q = \frac{\nu}{H}\rho_m\rho_d$, ($\nu > 0$) and iii) $Q = 3\nu H\rho_d$, ($\nu > 0$). The first choice of the interaction term is a particular case of a general linear combination of the energy densities while the second choice is physically more viable in the sense that interaction rate vanishes if one of the energy densities is zero and increases with each of the densities.

III. HOLOGRAPHIC DE MODEL WITH EVENT HORIZON AS IR CUT OFF

The radius of the event horizon is defined by the improper integral

$$R_E = a \int_t^\infty \frac{dt}{a} \quad (9)$$

Note that above improper integral converges only when strong energy condition is violated. So in the present accelerating phase it always exists. Now choosing $L = R_E$, we have from (1)

$$\rho_d = \frac{3M_p^2 c^2}{R_E^2} \quad (10)$$

From the conservation equations (7) and (8) and using the expression for energy density ρ_d from equation (10), the equation of state parameter takes the following form for any interaction Q ,

$$\omega_d = -\frac{1}{3} - \frac{2\sqrt{\Omega_d}}{3c} - \frac{Q}{3H\rho_d} \quad (11)$$

Now the second Friedmann equation (3) and the energy conservation for DE (i.e equation (8)) can be reduced (after a bit simplification) into an autonomous system as

$$\dot{H} = -\frac{3H^2}{2} \left[1 - \frac{\Omega_d}{3} - \frac{2\Omega_d^{\frac{3}{2}}}{3c} - \frac{Q}{3H\rho} \right] \quad (12)$$

$$\dot{\rho}_d = 2\rho_d H \left[\frac{\sqrt{\Omega_d}}{c} - 1 \right] \quad (13)$$

Now in (ρ_d, H) phase plane this autonomous system has a line of critical points for $\sqrt{\Omega_d} = c$ if

$$Q = 3(1 - c^2)H\rho \quad (14)$$

We shall now analyze the evolution equations for different choices of the interaction term separately.

$$\mathbf{A.} \quad Q = 3b^2 H \rho = 3b^2 H(\rho_m + \rho_d)$$

For the interaction, $Q = 3b^2 H \rho$, the above autonomous system becomes

$$\dot{H} = -\frac{1}{2} \left[3H^2(1 - b^2) - \frac{\rho_d}{3} - \frac{2\rho_d^{\frac{3}{2}}}{3\sqrt{3}cH} \right] \quad (15)$$

$$\dot{\rho}_d = 2\rho_d H \left[\frac{\sqrt{\Omega_d}}{c} - 1 \right] \quad (16)$$

In the (ρ_d, H) phase plane the autonomous system (15) and (16) has a line of critical points along the parabola $\rho_d = 3c^2 H^2$, provided $b^2 = 1 - c^2$ from the above discussion. Critical points are given in

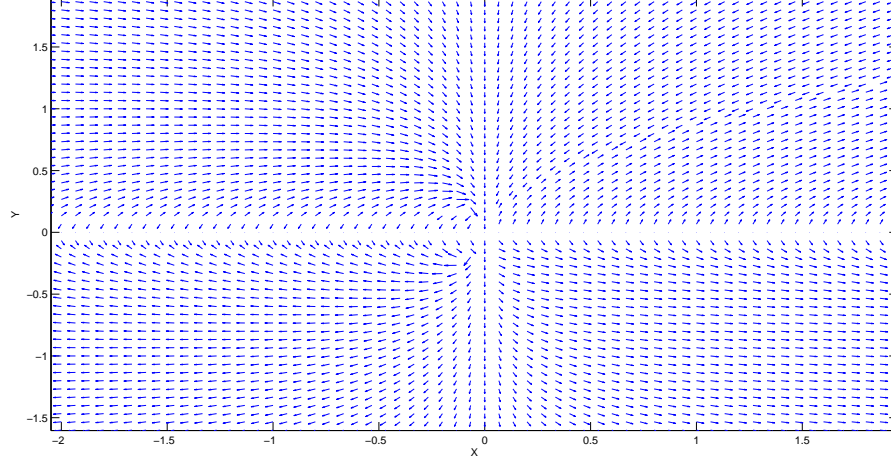


FIG. 1. Direction field for the autonomous system (15) and (16) for $c = 0.65$

TABLE I. *Equilibrium points, their nature for $Q = 3b^2H\rho$*

Equilibrium point	ρ_d	H	b^2	Ω_d	ω_d	Nature
E_1	$3c^2H^2$	H	$1 - c^2$	c^2	< -1	hyperbolic: saddle
E_2	0	H	1	0	$< -\frac{1}{3}$	non-hyperbolic

table I. Direction field for the autonomous system given by equation (15) and (16) is drawn in Fig 1 for $c = 0.65$. Miao Li et al studied constraints on HDE model using Planck data [44]. The value of c is chosen here considering the constraints on 'c' from Planck data. We can see that there is no stable critical point i.e no attractor solution exists for the system. The determinant of linearized matrix for the autonomous system along the parabola is $-8c^2H^2 < 0$ representing the saddle type nature. The value of the cosmological parameters namely $\Omega_d = c^2$, $\omega_d < -\frac{1}{c^2} < -1$ and $u = \frac{b^2}{c^2}$ hold along the line of critical points. So we say that along the phase paths the ratio of the energy densities bears a constant value partially solving coincidence problem and the universe will be in the phantom era representing accelerating universe. E_1 is one of those points lying on the line of critical points. But these points are unstable. There may be another critical point for $\rho_d = 0$, $(0, H)$, H unspecified if $b^2 = 1$. But the critical point is non-hyperbolic in nature, the determinant being zero. Therefore, no definite information about stability can be extracted from this point, but it can describe a late time acceleration.

Further the conservation equations (4) and (5) can be expressed in the form

$$\dot{\rho}_m = \sqrt{3(\rho_m + \rho_d)}[b^2\rho_d - (1 - b^2)\rho_m] \quad (17)$$

TABLE II. *Equilibrium points, their nature for $Q = 3b^2 H \rho$*

Equilibrium point	ρ_d	ρ_m	ω_d	Nature
E_{1a}	ρ_d	$\frac{b^2}{1-b^2}\rho_d$	< -1	non-hyperbolic
E_{1b}	0	0	undefined	undefined

and

$$\dot{\rho}_d = -\sqrt{3(\rho_m + \rho_d)}[b^2\rho_m + (1 + \omega_d + b^2)\rho_d] \quad (18)$$

Note that above set of equations (17) and (18) will form an autonomous set provided the equation of state parameter for DE is constant. For the autonomous system $\dot{\rho}_m = 0$ and $\dot{\rho}_d = 0$, we get either $\rho_m = 0$, $\rho_d = 0$ or $\rho_m = \frac{b^2}{1-b^2}\rho_d$ provided $\omega_d = \frac{1}{b^2-1} < -1$ which supports cosmic acceleration.

We see that from equation (18), in the phantom domain ($\omega_d < -1$), ρ_d may begin to increase and dominate over dark matter. Thus the present model of the universe has dark energy dominance at early epoch and at late time accelerated scenario while dark matter dominates in the intermediate stages of the evolution.

$$\mathbf{B.} \quad Q = \frac{\nu}{H}\rho_m\rho_d, \quad (\nu > 0)$$

For this choice of interaction term, the equation of state parameter is

$$\omega_d = -\frac{1}{3} - \frac{2\sqrt{\Omega_d}}{3c} - \nu(1 - \Omega_d) \quad (19)$$

and

$$\dot{u} = 3Hu[-\frac{1}{3} - \frac{2}{3c\sqrt{1+u}} + \frac{\nu}{1+u}] \quad (20)$$

and the evolution of density parameter is modified as

$$\dot{\Omega}_d = H\Omega_d(1 - \Omega_d)[1 - 3\nu\Omega_d + \frac{2\sqrt{\Omega_d}}{c}] \quad (21)$$

The second Friedmann equation (3) can be expressed in terms of density parameter as

$$\dot{H} = -\frac{3H^2}{2}[1 - \frac{\Omega_d}{3} - \frac{2\Omega_d^{\frac{3}{2}}}{3c} - \nu\Omega_d(1 - \Omega_d)] \quad (22)$$

Here we have an autonomous system in the phase plane (Ω_d, H) corresponding to equations (21) and (22). In (Ω_d, H) phase plane this autonomous system has a line of critical points for $\sqrt{\Omega_d} = c$ if either $c^2 = 1$ or $\nu c^2 = 1$ and universe will be in phantom era. E_6 is one of those critical points. Fig 2 and Fig 3 presents the direction field and phase portrait for this autonomous system for $c = .62$ (Note

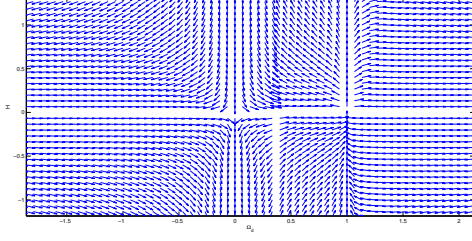


FIG. 2. direction field for the system (21) and (22) for $c = .62$

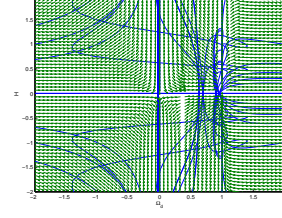


FIG. 3. phase portrait for the system (21) and (22) for $c = .62$

TABLE III. *Equilibrium points, their nature for $Q = \frac{\nu}{H}\rho_m\rho_d$*

Equilibrium point	Ω_d	H	c^2	ω_d	Nature
E_3	Ω_d	0		$< -\frac{1}{3}$	non-hyperbolic
E_4	0	0		$-\frac{1}{3} - \nu$	non-hyperbolic
E_5	1	0		$-\frac{1}{3} - \frac{2}{3c}$	non- hyperbolic
E_6	$c^2 \frac{1}{\nu c^2}$	1 or $\frac{1}{\nu}$		$-1 + \nu c^2 - \nu$	non-hyperbolic

that the whole region depicted in the figure is not viable). The possible critical points are E_3 ($H = 0$, Ω_d is unrestricted), E_4 (0, 0), E_5 (1, 0) and E_6 , $(c^2, \frac{1}{\nu c^2})$ if either $c^2 = 1$ or $\nu c^2 = 1$. All the critical points are listed in table III. The critical points are all non hyperbolic. So stability of these points are not conclusive.

In table III, the first three critical points with $H = 0$ represent static model of the universe. E_4 represents purely dark matter dominated universe while critical point E_5 corresponds to universe filled with dark energy only with $\omega_d = -\frac{1}{3} - \frac{2}{3c}$ corresponding to phantom domain of evolution of the universe. For the critical point E_6 , considering $\nu c^2 = 1$, $u = \frac{1-c^2}{c^2}$ and $\omega_d = -\nu$. This critical point will be in the dark energy dominated era or dark matter dominated era according as $c^2 > 1$ or $c^2 < 1$. We can see that all the points describe late time acceleration. If $\nu > 1$, then E_6 crosses the phantom barrier. If $c^2 = 1$, this point is in phantom barrier. We are unable to establish its stability, otherwise varying ν this critical point could describe present scenario of the universe.

Further the conservation equations (7) and (8) as before can be written as

$$\dot{\rho}_m = \rho_m \left[\frac{\nu}{H} \rho_d - \sqrt{3(\rho_m + \rho_d)} \right] \quad (23)$$

$$\dot{\rho}_d = -\rho_d \left[\frac{\nu}{H} \rho_m + (1 + \omega_d) \sqrt{3(\rho_m + \rho_d)} \right] \quad (24)$$

Here $\dot{\rho}_m = 0$ along the curve $\nu^2 \rho_d^2 = (\rho_m + \rho_d)^2$ or $\nu \Omega_d = 1$ and as a result both the energy components are of comparable magnitude i.e $u \sim o(1)$ twice during the evolution and give a possible explanation to the coincidence problem.

TABLE IV. *Equilibrium points, their nature for $Q = 3\nu H\rho_d, \nu > 0$*

Equilibrium point	Ω_d	H	ν	ω_d	Nature
E_7	c^2	H	$\frac{1-c^2}{c^2}$	$-\frac{1}{c^2}$	non-hyperbolic
E_8	Ω_d	0		$< -\frac{1}{3} - \nu$	non-hyperbolic
E_9	1	0		$< -\frac{1}{3} - \frac{2}{3c} - \nu$	non-hyperbolic
E_{10}	1	$H(\neq 0)$	$\frac{2}{3}(1 - \frac{1}{c})$	-1	non- hyperbolic

C. $Q = 3\nu H\rho_d$

In this case,

$$\dot{\Omega}_d = H\Omega_d(1 - \Omega_d)[1 - \frac{3\nu\Omega_d}{1 - \Omega_d} + \frac{2\sqrt{\Omega_d}}{c}] \quad (25)$$

The second Friedmann equation (3) can be expressed in terms of density parameter as

$$\dot{H} = -\frac{3H^2}{2}[1 - \frac{\Omega_d}{3} - \frac{2\Omega_d^{\frac{3}{2}}}{3c} - \nu\Omega_d] \quad (26)$$

Like previous section, (25) and (26) form an autonomous system in (Ω_d, H) phase plane . This autonomous system has a line of critical points for $\sqrt{\Omega_d} = c$ if $\nu = \frac{1-c^2}{c^2}$. E_7 represents one of those critical points. These critical points will be in the dark energy dominated era or dark matter dominated era according as $c^2 > 0$ or $c^2 < \frac{1}{2}$. We find that there are other critical points also namely, E_8 ($\Omega_d, 0$), Ω_d unrestricted, E_9 (1, 0), E_{10} (1, H) for $\nu = \frac{2}{3}(1 - \frac{1}{c})$, $H \neq 0$ (see table IV). The points describe late time acceleration, but all of them are non-hyperbolic critical points so could not infer the stability of these points. Like the previous section the critical points with $H = 0$ i.e E_8 and E_9 represent static model of the universe. E_{10} is interesting in the sense that it represents accelerating universe purely dominated by dark energy.

For the above three interactions sometimes we could not conclude on stability of critical points being non-hyperbolic, though this model solves coincidence problem partially.

IV. MODIFIED HOLOGRAPHIC RICCI DARK ENERGY MODEL

Here we associate infrared cutoff L with dark energy density which can be written as [45]

$$\rho_d = \frac{2}{\alpha - \beta}(\dot{H} + \frac{3\alpha}{2}H^2) \quad (27)$$

α, β being free constants. We have chosen α, β remembering the range preferred by authors [46,47] estimating the parameters in most cases. But to have a clear understanding, we have used several

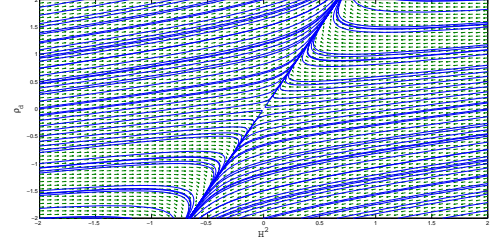
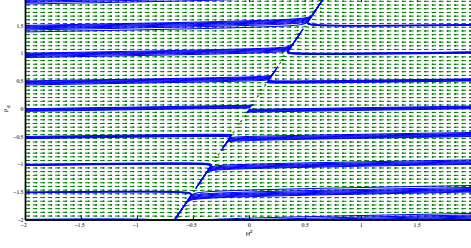


FIG. 4. phase portrait for the system (34) and (35) where $\alpha = 1.01, \beta = -0.01, b^2 = .01$

FIG. 5. phase portrait for the system (34) and (35) where $\alpha = 1.1, \beta = -.13, b = .01$

values in plotting phase portraits. The whole phase space may not be the feasible region considering cosmological constraints. Also the equation of state parameter for DE takes the form

$$\omega_d = -(\alpha - \beta) + \frac{\alpha}{\Omega_d} - \frac{1}{\Omega_d} \quad (28)$$

Using field equations, the deceleration parameter is given by

$$q = -\frac{\ddot{a}}{2H^2} = \frac{3}{2}\alpha - 1 - \frac{3}{2(1+u)}(\alpha - \beta) \quad (29)$$

We shall now analyze the dynamical equations for the three prescribed choices of the interactions.

$$\mathbf{A.} \quad Q = 3b^2 H \rho$$

Using field equations and the above form of the equation of state parameter, the energy conservation equations can be written explicitly in the form

$$\dot{\rho}_m = \sqrt{3(\rho_m + \rho_d)} [b^2 \rho_d - (1 - b^2) \rho_m] \quad (30)$$

and

$$\dot{\rho}_d = -\sqrt{3(\rho_m + \rho_d)} [b^2(\rho_m + \rho_d) + (1 + \omega_d)\rho_d] \quad (31)$$

So the evolution of the density parameter and u , the ratio of the energy densities are given by

$$\dot{\Omega}_d = -3H[-\Omega_d(1 - \Omega_d)(\alpha - \beta) + (\alpha - 1)(1 - \Omega_d) + b^2] \quad (32)$$

and

$$\dot{u} = 3H[-(\alpha - \beta)u + u(1 + u)(\alpha - 1) + b^2(1 + u)^2] \quad (33)$$

Equation (31) shows that ρ_d gradually decreases with the evolution of the universe till the universe enters the phantom era (i.e $\omega_d < -1$). Thus if DE density is assumed to be sufficiently large at early

TABLE V. *Equilibrium points, their nature for $Q = 3b^2 H\rho$*

Equilibrium point	H^2	ρ_d	q	Nature
E_{1a}	0	0	$\frac{3\beta}{2} - 1$	hyperbolic: saddle if $b^2 < \frac{\beta}{\beta-\alpha}$, stable if $b^2 > \frac{\beta}{\beta-\alpha}$

phases of the evolution then from equation (30) ρ_m increases till some intermediate stage when $\dot{\rho}_m = 0$ and then decreases. Here the transition of ρ_m occurs along the straight line $\rho_m = \frac{b^2}{1-b^2}\rho_d$ in the (ρ_m, ρ_d) plane. So similar to the previous section, it may be a possible resolution of the coincidence problem and the late time dominance by DE matches with present day observations. Moreover, choosing $x = \ln a$ as the time variable, the second Friedmann equation (3) and the conservation equation (8) can be formulated as a linear homogeneous autonomous system in the phase plane (ρ_d, H^2) :

$$(H^2)' = -3\alpha H^2 + (\alpha - \beta)\rho_d \quad (34)$$

$$\rho_d' = -9(\alpha - 1 + b^2)H^2 - 3\rho_d(1 - \alpha + \beta) \quad (35)$$

Here $(0, 0)$ i.e origin is the critical point (see table V). The determinant of the linearized matrix is $9(\alpha - \beta)b^2 + 9\beta$. If $b^2 < \frac{\beta}{\beta-\alpha}$ the critical point is a saddle point and if $b^2 > \frac{\beta}{\beta-\alpha}$ then it is stable but it can not be a viable cosmological solution. Nature of this critical point is presented in Fig 4 and Fig 5 considering some values for α and β . Fig 4 is drawn choosing the values of α, β as suggested by the authors [45] for this interaction considering the observational support.

$$\text{B. } Q = \frac{\nu}{H}\rho_m\rho_d, \nu > 0$$

The explicit form of the energy conservation equations are given by

$$\dot{\rho}_m = \rho_m\left(\frac{\nu}{H}\rho_d - 3H\right) \quad (36)$$

and

$$\dot{\rho}_d = -\rho_d\left[\frac{\nu}{H}\rho_m + 3H(1 + \omega_d)\right] \quad (37)$$

As a result the evolution of u (ratio of energy densities) obeys

$$\dot{u} = 3Hu[(\alpha - 1)(1 + u) - (\alpha - \beta) + \nu] \quad (38)$$

The second Friedmann equation (3) becomes

$$\dot{H} = -\frac{3H^2}{2}\left[\alpha - \frac{\alpha - \beta}{(1 + u)}\right] \quad (39)$$

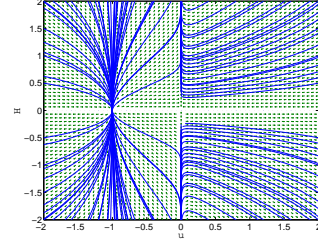
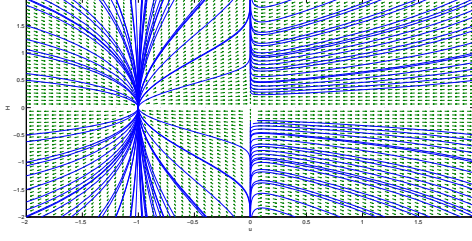


FIG. 6. phase portrait for the system (38) and (39) where $\alpha = 1.01, \beta = -0.01, \nu = .1$ FIG. 7. Phase portrait for the system (38) and (39) where $\alpha = 1.1, \beta = -.13, \nu = .1$

TABLE VI. *Equilibrium points, their nature for $Q = \frac{\nu}{H}\rho_m\rho_d$*

Equilibrium point	u	H	ν	q	Nature
E_{2b}	u	0		$\frac{3\alpha}{2} - 1 - \frac{3(\alpha-\beta)}{2(1+u)}$	non-hyperbolic
E_{3b}	0	0		$\frac{3\beta}{2} - 1$	non-hyperbolic
E_{4b}	$-\frac{\beta}{\alpha}$	H	$\frac{1-\beta}{\alpha}$	-1	non- hyperbolic

The equations (38) and (39) form an autonomous system in (u, H) plane. Phase portrait of this system is drawn in Fig 6 and Fig 7 for some α, β . There is a line of critical points for $H = 0$ but not of physical interest. Another critical point is $(-\frac{\beta}{\alpha}, H)$ where H is unspecified if $\alpha\nu + \beta - 1 = 0$. These points are listed in table VI. Again the critical points are non hyperbolic. From the figure we can see that no stable point exists. If β is negative, u is valid constant i.e energy densities of dark matter and dark energy somewhat comparable, then E_{4b} having present acceleration can be interesting.

C. $Q = 3\nu H\rho_d, \nu > 0$

In this case we consider the autonomous system in (u, H) plane like previous section

$$\dot{u} = 3Hu[(\alpha - 1)u - 1 + \beta] + \nu + \frac{\nu}{u} \quad (40)$$

$$\dot{H} = -\frac{3H^2}{2}\left[\alpha - \frac{\alpha - \beta}{(1 + u)}\right] \quad (41)$$

Phase portrait of this system consisting of (40) and (41) is drawn in Fig 8 and Fig 9. No stable attractor exists in the figure. Again, there is a line of critical points for $H = 0$ but not of physical interest. Another critical point may be $(-\frac{\beta}{\alpha}, H)$ provided $\nu = -\frac{\beta}{\alpha}$ (see table VII). We have taken negative value of β here to have valid ν for the point E_{7b} . These points are non hyperbolic in nature so local stability is not conclusive.

TABLE VII. *Equilibrium points, their nature for $Q = 3\nu H\rho_d$*

Equilibrium point	u	H	ν	q	Nature
E_{5b}	u	0		$\frac{3\alpha}{2} - 1 - \frac{3(\alpha-\beta)}{2(1+u)}$	non-hyperbolic
E_{6b}	0	0		$\frac{3\beta}{2} - 1$	non-hyperbolic
E_{7b}	$-\frac{\beta}{\alpha}$	H	$-\frac{\beta}{\alpha}$	-1	non- hyperbolic

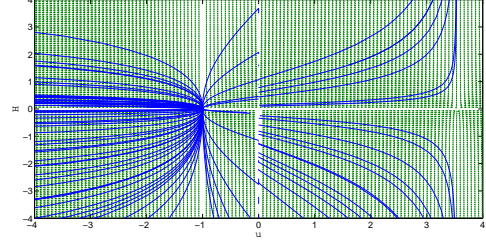
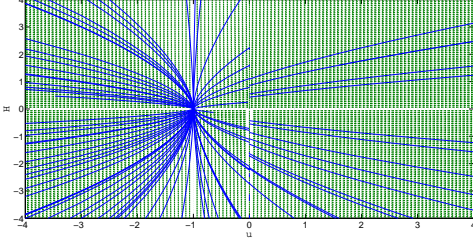


FIG. 8. phase portrait for the system (40) and (41) where $\alpha = 1.01, \beta = -0.01, \nu = .001$

FIG. 9. phase portrait for the system (40) and (41) where $\alpha = 1.3, \beta = -0.1, \nu = .001$

V. SUMMARY

The present work deals with the wellknown interacting holographic dark energy model from the point of view of dynamical system analysis. The cosmic fluid consists of two interacting dark sectors namely dark matter in the form of dust and (modified) holographic dark energy in the form of perfect fluid. The energy density and the variable equation of state parameter for the dark energy is determined from the holographic principle. The interaction terms between the two dark sectors are chosen phenomenologically but are widely used in literature. Here we have considered three choices of the interaction term to avoid mathematical complexity - i) proportional to the total energy density of the dark sectors, ii) proportional to the product of energy densities of dark matter and dark energy and iii) proportional to dark energy density. However, we have not considered interaction with baryonic matter due to constraints imposed by local gravity measurements. The future event horizon is chosen as IR cut off for holographic dark energy model while IR cut off for modified holographic dark energy is chosen at Ricci length scale. In both the models, with three choices of interaction, the evolution equations are suitably reduced to an autonomous system. In most of the cases, we have found line of critical points which are presented in tables. The nature of the hyperbolic critical points and their stability are analyzed using jacobian matrix of the dynamical system.

We can see from the tables that for the HDE model, three different interacting terms lead to same line of critical points $\Omega_d = c^2$ with different constraints on ν and c . An arbitrary point on this curve can cross the phantom barrier depending on values of ν and c . The points E_6, E_7 are among those

points. For MHRDE model E_{4b} , E_{7b} are interesting as they describe accelerating universe. But we could not conclude on stability of the above points due to their non-hyperbolic nature.

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